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LETTER TO THE EDITOR

Ultradiscretization of elliptic functions and its applications to integrable systems

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Abstract
It is shown that there exist three kinds of ultradiscrete analogues of Jacobi's elliptic functions. In this process, the asymptotic behaviour of the poles and the zeros of the functions plays a crucial role. Using the ultradiscrete analogues and an addition formula, exact solutions to the ultradiscrete KP equation are constructed, and their relation to the ultradiscrete QRT system is discussed.

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1. Introduction

An ultradiscrete system is a dynamical system in which both independent and dependent variables take discrete values. One of the most famous ultradiscrete systems is the box–ball system (BBS) in which finitely many balls move in an array of boxes under a certain time evolution rule [1, 2]. The BBS is obtained from a discrete analogue of the Lotka–Volterra equation [3] through a limiting procedure called ultradiscretization. The procedure of ultradiscretization was introduced by Tokihiro *et al* [4] in order to establish a direct connection between the KdV equation and the soliton cellular automaton proposed by Takahashi and Satsuma [5]. A remarkable feature in the procedure of ultradiscretization is that if we ultradiscretize an exact solution to a discrete system, then it solves the corresponding ultradiscrete one. In fact, the N-soliton solution to the BBS is directly obtained from the one to the Lotka–Volterra equation through the ultradiscretization.

The procedure of ultradiscretization can be applied to some exact solutions other than the soliton solutions. In [6], Takahashi *et al* presented the elliptic function solutions to the autonomous limit of the ultradiscrete Painlevé I equation through the ultradiscretization. In order to apply the procedure of ultradiscretization to Jacobi's elliptic functions, they carefully related the nome of elliptic theta function to a positive parameter ε and derived asymptotic forms of $\mathrm{sn}^2 u$ and $\mathrm{cn}^2 u$ with respect to ε. Then, by taking the limit $\varepsilon \to +0$, they obtained the ultradiscrete analogues of $\mathrm{sn}^2 u$ and $\mathrm{cn}^2 u$. In this letter, we describe how the relation between the nome of elliptic theta function and the parameter ε dominates the ultradiscrete analogues of Jacobi's elliptic functions. The asymptotic behaviour of the zeros of elliptic theta function plays a crucial role in this process. Consequently, we obtain three kinds of ultradiscrete analogues of Jacobi's elliptic functions, which are classified by the limiting value of the modulus in the limit $\varepsilon \to +0$. Finally, we construct cnoidal wave solutions to the ultradiscrete KP equation by using the ultradiscrete analogue of $\operatorname{sn} u$. Then we discuss their relation to the ultradiscrete QRT system.

2. Ultradiscretization of an elliptic theta function

The elliptic theta function $\vartheta_{00}(z, \tau)$ is defined as follows [7]:

$$\vartheta_{00}(z, \tau) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z},$$

where $\tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $z \in \mathbb{C}$. Remark that $\vartheta_{00}(z, \tau)$ has the zeros at

$$z = \frac{1}{2} + \frac{\tau}{2} + m + n\tau \qquad (m, n \in \mathbb{Z}). \tag{1}$$

By using Jacobi's transformation formula,

$$\vartheta_{00}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = e^{-\frac{\pi i}{4}} \tau^{\frac{1}{2}} e^{\frac{\pi i z^2}{\tau}} \vartheta_{00}(z, \tau),$$

$\vartheta_{00}(z, \tau)$ reduces to

$$\vartheta_{00}(z, \tau) = e^{\frac{\pi i}{4}} \tau^{-\frac{1}{2}} e^{-\frac{\pi i z^2}{\tau}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n^2}{\tau}} e^{\frac{2\pi i n z}{\tau}}.$$

Let $\tau = i\pi\varepsilon$, where ε is a positive number. Also let us denote

$$\log^{\dagger} f_1(z) f_2(z) \cdots f_n(z) := \operatorname{Log} f_1(z) + \operatorname{Log} f_2(z) + \cdots + \operatorname{Log} f_n(z),$$

where $\operatorname{Log} f_j(z)$ stands for the principal value of $\log f_j(z)$. Then we have

$$\varepsilon \log^{\dagger} \vartheta_{00}(x, i\pi\varepsilon) = \varepsilon \operatorname{Log}(\pi\varepsilon)^{-\frac{1}{2}} + \varepsilon \operatorname{Log} e^{-\frac{x^2}{\varepsilon}} + \varepsilon \operatorname{Log} \sum_{n=-\infty}^{\infty} e^{\frac{2nx - n^2}{\varepsilon}},$$

where $x = \operatorname{Re} z \in \mathbb{R}$. Note that the imaginary part vanishes. For any $N \in \mathbb{N}$, the function $\varepsilon \operatorname{Log} \sum_{n=-N}^{N} \exp[(2nx - n^2)/\varepsilon]$ uniformly converges to $\max_{n=-N}^{N}[2nx - n^2]$ in the limit $\varepsilon \to +0$. Moreover, this uniformly converges to $\max_{n=-\infty}^{\infty}[2nx - n^2]$ on $|x| < N + 1/2$ in the limit $N \to \infty$. Hence, we can take the limit $\varepsilon \to +0$ and obtain a piecewise quadratic function denoted by $\Theta_0(x)$:

$$\lim_{\varepsilon \to +0} \varepsilon \log^{\dagger} \vartheta_{00}(x, i\pi\varepsilon) = -x^2 + \max_{n=-\infty}^{\infty} [2nx - n^2] =: \Theta_0(x).$$

Remark 1. The function $\Theta_0(x)$ takes the local minimum at $x = 1/2 + m (m \in \mathbb{Z})$ which are the limiting points of (1), the zeros of $\vartheta_{00}(z, \tau)$, in the limit $\varepsilon \to +0$.

Remark 2. The infinite product expansion for $\vartheta_{00}(z, \tau)$ leads to another expression:

$$\Theta_0(x) = -x^2 + \sum_{n=1}^{\infty} \max[0, |2x| - 2n + 1].$$

An addition formula for $\Theta_0(x)$ should be noted [8]; for any $x, u \in \mathbb{R}$, we have

$$\Theta_0(x + u) + \Theta_0(x - u) = 2\max\left[\Theta_0(x) + \Theta_0(u), \Theta_{\frac{1}{2}}(x) + \Theta_{\frac{1}{2}}(u)\right], \tag{2}$$

where we put $\Theta_{\frac{1}{2}}(x) := \Theta_0(x + 1/2)$.

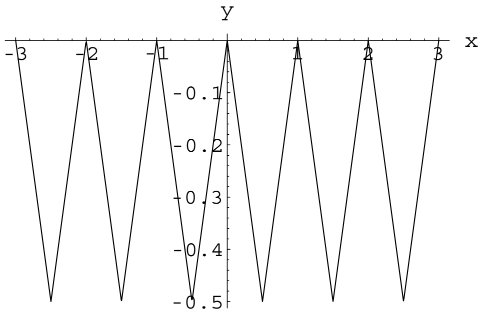

Figure 1. The ultradiscrete analogue of $\mathrm{dn}\, u$, $y = -J(x)$, with $\tau = \mathrm{i}\pi\varepsilon$.

3. Ultradiscrete analogues of Jacobi's elliptic functions

Jacobi's elliptic functions $\mathrm{sn}\, u$, $\mathrm{cn}\, u$ and $\mathrm{dn}\, u$ are ratios of elliptic theta functions:

$$\mathrm{sn}\, u := -\frac{\vartheta_{00}(0,\tau)}{\vartheta_{10}(0,\tau)}\frac{\vartheta_{11}(z,\tau)}{\vartheta_{01}(z,\tau)}$$

$$\mathrm{cn}\, u := \frac{\vartheta_{01}(0,\tau)}{\vartheta_{10}(0,\tau)}\frac{\vartheta_{10}(z,\tau)}{\vartheta_{01}(z,\tau)}$$

$$\mathrm{dn}\, u := \frac{\vartheta_{01}(0,\tau)}{\vartheta_{00}(0,\tau)}\frac{\vartheta_{00}(z,\tau)}{\vartheta_{01}(z,\tau)}$$

where we put $u := \pi\vartheta_{00}^2(0,\tau)z$. Therefore, their ultradiscrete analogues are ruled by the asymptotic behaviour of the zeros of elliptic theta functions in the limit $\varepsilon \to +0$, which depends only on the relation between τ and ε (see remark 1).

As an example, we consider the parametrization $\tau = \mathrm{i}\pi\varepsilon$ as in the previous section. Since $\vartheta_{01}(z,\tau)$ is just the translation $z \mapsto z + 1/2$ of $\vartheta_{00}(z,\tau)$, we have

$$\lim_{\varepsilon\to+0} \varepsilon \log^\dagger \mathrm{dn}\, u = \lim_{\varepsilon\to+0} \varepsilon \log^\dagger \frac{\vartheta_{01}(0,\mathrm{i}\pi\varepsilon)}{\vartheta_{00}(0,\mathrm{i}\pi\varepsilon)}\frac{\vartheta_{00}(x,\mathrm{i}\pi\varepsilon)}{\vartheta_{01}(x,\mathrm{i}\pi\varepsilon)} = \Theta_{\frac{1}{2}}(0) + \Theta_0(x) - \Theta_{\frac{1}{2}}(x)\,.$$

The resulting function, which is denoted by $-J(x)$, is piecewise linear and periodic with period 1 and takes the local maximum (resp. the local minimum) at $z = m$ (resp. $z = m + 1/2$) ($m \in \mathbb{Z}$), which are the limiting points of the poles (resp. the zeros) of $\mathrm{dn}\, u$ in the limit $\varepsilon \to +0$ (see figure 1).

Since $\vartheta_{10}(z,\tau)$ and $\vartheta_{11}(z,\tau)$ are the translations $z \mapsto z + \tau/2$ and $z \mapsto z + 1/2 + \tau/2$ of $\vartheta_{00}(z,\tau)$ respectively, $\mathrm{sn}\, u$ and $\mathrm{cn}\, u$ have the zeros on the real axis. Therefore, we take an appropriate line parallel to the real axis and apply the procedure of ultradiscretization on it. Then we obtain the ultradiscrete analogues:

$$\begin{aligned}\lim_{\varepsilon\to+0} \varepsilon \log^\dagger \mathrm{sn}\, u|_{\mathrm{Im}\, z=\frac{\mathrm{i}\pi}{4}\varepsilon} &= \lim_{\varepsilon\to+0} \varepsilon \log^\dagger \left[-\frac{\vartheta_{00}(0,\mathrm{i}\pi\varepsilon)}{\vartheta_{10}(0,\mathrm{i}\pi\varepsilon)}\frac{\vartheta_{11}(z,\mathrm{i}\pi\varepsilon)}{\vartheta_{01}(z,\mathrm{i}\pi\varepsilon)}\right]\Bigg|_{\mathrm{Im}\, z=\frac{\mathrm{i}\pi}{4}\varepsilon} \\ &= 0 \\ \lim_{\varepsilon\to+0} \varepsilon \log^\dagger \mathrm{cn}\, u|_{\mathrm{Im}\, z=\frac{\mathrm{i}\pi}{4}\varepsilon} &= \lim_{\varepsilon\to+0} \varepsilon \log^\dagger \frac{\vartheta_{01}(0,\mathrm{i}\pi\varepsilon)}{\vartheta_{10}(0,\mathrm{i}\pi\varepsilon)}\frac{\vartheta_{10}(z,\mathrm{i}\pi\varepsilon)}{\vartheta_{01}(z,\mathrm{i}\pi\varepsilon)}\Bigg|_{\mathrm{Im}\, z=\frac{\mathrm{i}\pi}{4}\varepsilon} \\ &= -J(x),\end{aligned}$$

where $x = \mathrm{Re}\, z$. The function $\mathrm{sn}\, u$ has the poles at $z = m + (n + 1/2)\,\tau$ and the zeros at $z = m + n\tau (m, n \in \mathbb{Z})$; hence they converge to the same points $x = m$ in the limit $\varepsilon \to +0$.

Table 1. The ultradiscrete analogues of elliptic theta functions. We assume $\tau \to p/q$ ($\varepsilon \to +0$) and put $x = \mathrm{Re}\, z$.

Original	Ultradiscrete		
	p even q odd	p odd q even	p odd q odd
$\vartheta_{00}(z,\tau)$	$\Theta_0(qx)$	$\Theta_0(qx)$	$\Theta_{\frac{1}{2}}(qx)$
$\vartheta_{01}(z,\tau)$	$\Theta_{\frac{1}{2}}(qx)$	$\Theta_0(qx)$	$\Theta_0(qx)$
$\vartheta_{10}(z,\tau)$	$\Theta_0(qx)$	$\Theta_{\frac{1}{2}}(qx)$	$\Theta_0(qx)$
$\vartheta_{11}(z,\tau)$	$\Theta_{\frac{1}{2}}(qx)$	$\Theta_{\frac{1}{2}}(qx)$	$\Theta_{\frac{1}{2}}(qx)$

Therefore, the ultradiscrete analogue is a constant function. Although the process of applying the procedure of ultradiscretization is different from [6], these ultradiscrete analogues are exactly the same as the ones obtained in [6].

Now we consider the general case. Let us denote the sets of the zeros of $\mathrm{sn}\, u$, $\mathrm{cn}\, u$ and $\mathrm{dn}\, u$, i.e., that of $\vartheta_{11}(z,\tau)$, $\vartheta_{10}(z,\tau)$ and $\vartheta_{00}(z,\tau)$, by Z_s, Z_c and Z_d, respectively:

$$\begin{aligned} \mathrm{sn}\, u:&\quad Z_s = \{m_s + n_s\tau \mid m_s, n_s \in \mathbb{Z}\}\\ \mathrm{cn}\, u:&\quad Z_c = \left\{\left(m_c + \frac{1}{2}\right) + n_c\tau \mid m_c, n_c \in \mathbb{Z}\right\}\\ \mathrm{dn}\, u:&\quad Z_d = \left\{\left(m_d + \frac{1}{2}\right) + \left(n_d + \frac{1}{2}\right)\tau \mid m_d, n_d \in \mathbb{Z}\right\}. \end{aligned}$$

The set P of the poles, namely the set of the zeros of $\vartheta_{01}(z,\tau)$, is

$$P = \left\{m_p + \left(n_p + \frac{1}{2}\right)\tau \,\middle|\, m_p, n_p \in \mathbb{Z}\right\}.$$

If we assume $\tau \to p/q \in \mathbb{Q}$, then each two of the above four sets coincide respectively:

$$\begin{cases} Z_s = P \text{ and } Z_c = Z_d & (p \text{ even and } q \text{ odd})\\ Z_s = Z_c \text{ and } Z_d = P & (p \text{ odd and } q \text{ even})\\ Z_s = Z_d \text{ and } Z_c = P & (p \text{ odd and } q \text{ odd}), \end{cases} \tag{3}$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ are mutually disjoint. Let $\tau' = (a\tau + b)/(c\tau + d)$. Then, by putting $\tau = \mathrm{i}\pi\varepsilon$, $b = p$ and $d = q$, we have $\tau' \to p/q$ in the limit $\varepsilon \to +0$. Therefore, the above fact (3) suggests that two of the three functions reduce to singly-periodic ones, which are the same up to a constant, and the remaining to a constant in the limit $\varepsilon \to +0$.

Consider a functional equation [7]:

$$\vartheta_{00}\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \zeta(c\tau + d)^{\frac{1}{2}}\, \mathrm{e}^{\frac{\pi \mathrm{i} c z^2}{c\tau + d}}\, \vartheta_{00}(z,\tau), \tag{4}$$

where we assume $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$, $ab, cd \in 2\mathbb{Z}$ and ζ to be an eighth root of 1. Assume p even and q odd. Then we obtain

$$\lim_{\varepsilon \to +0} \varepsilon \log^{\dagger} \vartheta_{00}\left(\frac{x}{c\mathrm{i}\pi\varepsilon + q}, \frac{a\mathrm{i}\pi\varepsilon + p}{c\mathrm{i}\pi\varepsilon + q}\right) = \Theta_0(x).$$

Thus the ultradiscrete analogue of $\vartheta_{00}(z,\tau)$ with $\tau = (a\mathrm{i}\pi\varepsilon + p)/(c\mathrm{i}\pi\varepsilon + q) \to p/q$ is $\Theta_0(qx)$. In other cases, the ultradiscrete analogues are similarly obtained (see table 1).

Since Jacobi's elliptic functions are the ratios of elliptic theta functions, their ultradiscrete analogues are differences of that of elliptic theta functions, which are classified by the limiting value of the modulus $\kappa(\tau) := \vartheta_{10}^2(0,\tau)\big/\vartheta_{00}^2(0,\tau)$ in the limit $\varepsilon \to +0$ (see table 2). We see that the above estimation based on the behaviour of the poles and the zeros consists certainly.

Table 2. The ultradiscrete analogues of Jacobi's elliptic functions. We assume $\tau \to p/q$ ($\varepsilon \to +0$) and put $x = \mathrm{Re}\, z$ and $J(x) := \Theta_{\frac{1}{2}}(x) - \Theta_0(x) - \Theta_{\frac{1}{2}}(0)$.

	Ultradiscrete		
Original	p even q odd	p odd q even	p odd q odd
$\mathrm{sn}\, u$	0	$J(qx)$	$J(qx) + 2\Theta_{\frac{1}{2}}(0)$
$\mathrm{cn}\, u$	$-J(qx)$	$J(qx)$	0
$\mathrm{dn}\, u$	$-J(qx)$	0	$J(qx)$
	$\kappa(\tau) \to 1$	$\kappa(\tau) \to 0$	$\kappa(\tau) \to \infty$

4. Cnoidal wave solutions to the ultradiscrete KP equation and their relation to the ultradiscrete QRT system

Consider the bilinear form of the discrete KP equation [9]:

$$a_1(a_2 - a_3)\rho^t_{s,u+1}\rho^{t+1}_{s+1,u} + a_2(a_3 - a_1)\rho^{t+1}_{s,u}\rho^t_{s+1,u+1} + a_3(a_1 - a_2)\rho^t_{s+1,u}\rho^{t+1}_{s,u+1} = 0.$$

Employ a transformation $\rho^t_{s,u} \to \rho^t_{s-t,u+t+1} =: \rho^m_{n,l-1}$; then we obtain

$$a_1(a_2 - a_3)\rho^{m+1}_{n,l}\rho^m_{n,l} = a_2(a_1 - a_3)\left(\rho^m_{n+1,l}\right)^2 + a_3(a_2 - a_1)\left(\rho^m_{n+1,l-1}\right)^2, \tag{5}$$

where we assume $\rho^m_{n+1,l} = \rho^{m+1}_{n-1,l}$ and $\rho^m_{n,l-1} = \rho^m_{n,l+1}$. Compare (5) with an addition formula for elliptic theta functions

$$\vartheta_{01}(x+u,\tau)\vartheta_{01}(x-u,\tau)\vartheta^2_{01}(0,\tau) = \vartheta^2_{01}(x,\tau)\vartheta^2_{01}(u,\tau) - \vartheta^2_{11}(x,\tau)\vartheta^2_{11}(u,\tau). \tag{6}$$

Then, it is easy to see that

$$\rho^m_{n,l} = \mathrm{e}^{\pi\mathrm{i}\left(\frac{l}{2}\right)^2\tau}\,\mathrm{e}^{2\pi\mathrm{i}\frac{l}{2}\left(z+\frac{1}{2}\right)}\vartheta_{00}\left(z + \frac{1}{2} + \frac{l}{2}\tau, \tau\right), \qquad z = \eta(n+2m), \tag{7}$$

solves the discrete KP equation (5) with the choice of τ and η:

$$\frac{a_3(a_2 - a_1)}{a_2(a_1 - a_3)} = -\frac{\vartheta^2_{11}(\eta,\tau)}{\vartheta^2_{01}(\eta,\tau)}. \tag{8}$$

By employing the condition

$$\rho^m_{n,l} = \begin{cases} g^m_n & (l = 0 \ (\mathrm{mod}\ 2)) \\ f^m_n & (l = 1 \ (\mathrm{mod}\ 2)), \end{cases} \tag{9}$$

which is consistent with $\rho^m_{n,l-1} = \rho^m_{n,l+1}$, we have

$$a_1(a_2 - a_3)f^{m+1}_n f^m_n = a_2(a_1 - a_3)\left(f^m_{n+1}\right)^2 + a_3(a_2 - a_1)\left(g^m_{n+1}\right)^2 \tag{10a}$$

$$a_1(a_2 - a_3)g^{m+1}_n g^m_n = a_2(a_1 - a_3)\left(g^m_{n+1}\right)^2 + a_3(a_2 - a_1)\left(f^m_{n+1}\right)^2. \tag{10b}$$

Let

$$v = \frac{f^m_n}{g^m_n} \qquad w = \frac{f^m_{n+1}}{g^m_{n+1}} \qquad \bar{v} = \frac{f^{m+1}_n}{g^{m+1}_n} \qquad \bar{w} = \frac{f^{m+1}_{n+1}}{g^{m+1}_{n+1}}.$$

Then we obtain the following QRT mapping ${}^t(v, w) \mapsto {}^t(\bar{v}, \bar{w})$ [10]:

$$\bar{v}v = \frac{a_{13}w^2 + a_{11}}{a_{11}w^2 + a_{13}} \qquad \bar{w}w = \frac{a_{13}\bar{v}^2 + a_{11}}{a_{11}\bar{v}^2 + a_{13}}, \tag{11}$$

where we put

$$a_{11} := a_3(a_2 - a_1) \qquad a_{13} := a_2(a_1 - a_3).$$

Since (7) satisfies (9) and $\rho^m_{n+1,l} = \rho^{m+1}_{n-1,l}$, if (8) holds then

$$f^m_n = \vartheta_{11}(z, \tau), \qquad g^m_n = \vartheta_{01}(z, \tau), \qquad z = \eta(n + 2m), \tag{12}$$

solve (10*a*) and (10*b*). Therefore, the following solves (11):

$$\begin{cases} v = \dfrac{f^m_n}{g^m_n} = \dfrac{\vartheta_{11}(z, \tau)}{\vartheta_{01}(z, \tau)} = -\sqrt{\kappa(\tau)}\,\mathrm{sn}\,u \\ w = \dfrac{f^m_{n+1}}{g^m_{n+1}} = \dfrac{\vartheta_{11}(z + \eta, \tau)}{\vartheta_{01}(z + \eta, \tau)} = -\sqrt{\kappa(\tau)}\,\mathrm{sn}\,(u + \xi), \end{cases} \tag{13}$$

where we put $u := \pi\vartheta^2_{00}(0, \tau)z$ and $\xi := \pi\vartheta^2_{00}(0, \tau)\eta$. The invariant curve of (11) for the conserved quantity k is given as follows

$$a_{11}v^2w^2 + a_{13}v^2 + a_{13}w^2 + a_{11} - kvw = 0. \tag{14}$$

Substitute (12) into (14); then we have

$$\frac{\vartheta^2_{10}(0, \tau)}{\vartheta^2_{00}(0, \tau)} + \frac{\vartheta^2_{00}(0, \tau)}{\vartheta^2_{10}(0, \tau)} = \frac{k^2 - 4a^2_{11} - 4a^2_{13}}{4a_{11}a_{13}}. \tag{15}$$

We choose τ so as to satisfy (15). We can choose the sign of η so that

$$\frac{\vartheta^2_{01}(0, \tau)}{\vartheta_{00}(0, \tau)\,\vartheta_{10}(0, \tau)} \frac{\vartheta_{00}(\eta, \tau)\,\vartheta_{10}(\eta, \tau)}{\vartheta^2_{11}(\eta, \tau)} = \frac{k}{2a_{11}}. \tag{16}$$

Now we ultradiscretize the QRT mapping (11) and the solution (13). Replace the variables and the parameters with

$$v = \mathrm{e}^{\frac{V}{\varepsilon}} \qquad w = \mathrm{e}^{\frac{W}{\varepsilon}} \qquad \bar{v} = \mathrm{e}^{\frac{\bar{V}}{\varepsilon}} \qquad \bar{w} = \mathrm{e}^{\frac{\bar{W}}{\varepsilon}}$$
$$a_3(a_2 - a_1) = \mathrm{e}^{\frac{A_{11}}{\varepsilon}} \qquad a_2(a_1 - a_3) = \mathrm{e}^{\frac{A_{13}}{\varepsilon}} \qquad a_1(a_2 - a_3) = \mathrm{e}^{\frac{A_0}{\varepsilon}},$$

where ε is a positive number. Then, by taking the limit $\varepsilon \to +0$, we obtain a piecewise linear mapping ${}^t(V, W) \mapsto {}^t(\bar{V}, \bar{W})$ as the ultradiscrete analogue of (11):

$$\begin{cases} \bar{V} = -V + \max[2W + A_{13}, A_{11}] - \max[2W + A_{11}, A_{13}] \\ \bar{W} = -W + \max[2\bar{V} + A_{13}, A_{11}] - \max[2\bar{V} + A_{11}, A_{13}]. \end{cases} \tag{17}$$

Also replace the variables with $f^m_n = \exp\left(F^m_n/\varepsilon\right)$ and $g^m_n = \exp\left(G^m_n/\varepsilon\right)$ and take the limit $\varepsilon \to +0$; then we obtain the ultradiscrete analogue of (10*a*) and (10*b*):

$$F^{m+1}_n + F^m_n + A_0 = \max\left[2F^m_{n+1} + A_{13}, 2G^m_{n+1} + A_{11}\right] \tag{18a}$$

$$G^{m+1}_n + G^m_n + A_0 = \max\left[2G^m_{n+1} + A_{13}, 2F^m_{n+1} + A_{11}\right]. \tag{18b}$$

Under the condition

$$P^m_{n,l} = \begin{cases} G^m_n & (l = 0 \ (\mathrm{mod}\ 2)) \\ F^m_n & (l = 1 \ (\mathrm{mod}\ 2)), \end{cases} \tag{19}$$

which is the ultradiscrete analogue of (9), (18*a*) and (18*b*) are reduced from the ultradiscrete analogue of (5):

$$P^{m+1}_{n,l} + P^m_{n,l} + A_0 = \max\left[2P^m_{n+1,l} + A_{13}, 2P^m_{n+1,l-1} + A_{11}\right] \tag{20}$$

where we put $\rho^m_{n,l} = \exp\left(P^m_{n,l}/\varepsilon\right)$ and take the limit $\varepsilon \to +0$.

Let us ultradiscretize the solution (7). In order to obtain a nontrivial solution, substitute $(1+\mathrm{i}\pi\varepsilon/\lambda)/q$ ($\lambda \in \mathbb{R}, q \in \mathbb{N}$) into τ so that $\tau \to 1/q$ in the limit $\varepsilon \to +0$. (The following discussion does not depend on the parity of q.) Then we get

$$P^m_{n,l} = \lambda\Theta_0\left(qx + \frac{l}{2}\right) \qquad x = \mu(n+2m). \tag{21}$$

Note that (21) is consistent with $P^m_{n+1,l} = P^{m+1}_{n-1,l}$ and $P^m_{n,l+1} = P^m_{n,l-1}$. Compare (20) with the addition formula (2), which is the ultradiscrete analogue of (6). Then we see that (21) solves (20) if the following holds:

$$A_{11} - A_{13} = 2\lambda\Theta_{\frac{1}{2}}(q\mu) - 2\lambda\Theta_0(q\mu), \tag{22}$$

which is the ultradiscrete analogue of (8). Hence,

$$\begin{cases} V = F^m_n - G^m_n = \lambda J(qx) + \lambda\Theta_{\frac{1}{2}}(0) \\ W = F^m_{n+1} - G^m_{n+1} = \lambda J(qx + q\mu) + \lambda\Theta_{\frac{1}{2}}(0) \end{cases}$$

solve the ultradiscrete analogue (17) of the QRT mapping (11). Replace the conserved quantity k with $k = \exp(K/\varepsilon)$ and take the limit $\varepsilon \to +0$; then the invariant curve (14) reduces to the invariant curve of (17):

$$K = \max[A_{11} + V + W, A_{13} + V - W, A_{13} - V + W, A_{11} - V - W].$$

Noting (14), we have

$$k \geqslant 2a_{11} + 2a_{13} \qquad \text{for} \quad v > 0 \text{ and } w > 0.$$

Then (15) and (16) are ultradiscretized respectively as follows:

$$\begin{aligned} &-2\lambda\Theta_{\frac{1}{2}}(0) = 2K - A_{13} - A_{11} \\ &\lambda\left\{\Theta_0(q\mu) - \Theta_{\frac{1}{2}}(q\mu) - \Theta_{\frac{1}{2}}(0)\right\} = K - A_{11}. \end{aligned} \tag{23}$$

These are compatible with (22). Solving (22) and (23), q, λ and η are determined.

Finally, we comment on the relation between the ultradiscrete QRT system and tropical elliptic curves [11, 12]. A tropical elliptic curve is a smooth tropical algebraic variety of degree 3 and genus 1, and a tropical algebraic variety is defined by a piecewise linear function. On the other hand, the invariant curve of the QRT mapping is an elliptic curve and its ultradiscrete analogue is a polygon given by a piecewise linear function. However, the ultradiscrete analogue of an elliptic curve is not a tropical elliptic curve itself but a smaller part of it, namely the complement of the tentacles, on which the curve has a group structure in analogy with classical elliptic curves [11]. We shall report on a geometric description of the ultradiscrete QRT system in terms of the group structure of a tropical elliptic curve in a forthcoming paper.

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